

Mirror graphs: graph theoretical characterization of reflection arrangements and finite Coxeter groups

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Abstract

Mirror graphs were introduced by Brešar et al. in 2004 as an intriguing class of graphs: vertex-transitive, isometrically embeddable into hypercubes, having a strong connection with regular maps and polytope structure. In this article we settle the structure of mirror graphs by characterizing them as precisely the Cayley graphs of the finite Coxeter groups or equivalently the tope graphs of reflection arrangements – well understood and classified structures. We provide a polynomial algorithm for their recognition.

Keywords: mirror graphs, partial cubes, reflection arrangements, Cayley graphs, oriented matroids

1. Introduction and preliminaries

Brešar, Klavžar, Lipovec, and Mohar gave the following definition in [4]. Let G be a simple, connected graph. Call a partition $P = \{E_1, E_2, \dots, E_k\}$ of edges in G a *mirror partition* if for every $i \in \{1, \dots, k\}$, there exists an automorphism α_i of G such that:

- (i) for every edge $uv \in E_i$: $\alpha_i(u) = v$ and $\alpha_i(v) = u$
- (ii) $G - E_i$ consists of two connected components G_i^1 and G_i^2 , and α_i maps G_i^1 isomorphically onto G_i^2

A graph that has a mirror partition is called a *mirror graph*. By definition they are highly symmetrical graphs. In [4] it was shown that all mirror graphs are vertex-transitive, and certain connections with regular maps and polytope structures were established, indicating strong geometric properties of these graphs.

A more surprising result that they provided is that every mirror graph can be isometrically embedded (in the shortest path metric) into a hypercube graph Q_d , where Q_d is a graph whose vertices are vectors in $\{1, 0\}^d$ and two vertices are adjacent if they differ in exactly one coordinate. Graphs with this property are called *partial cubes*. In particular a mirror partition of edges in a mirror graph must coincide with the coordinate partition of the edges in the embedding. This implies that classes E_1, E_2, \dots, E_k can be recognized by the so called relation Θ on the edges of

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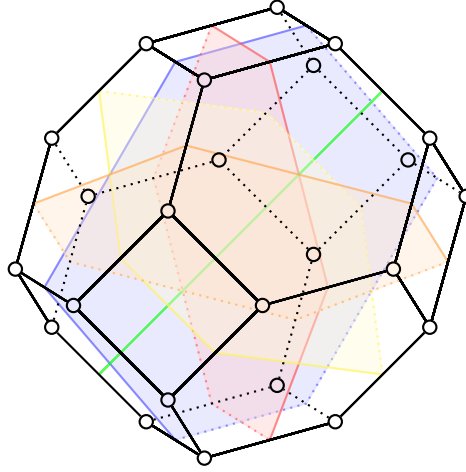


Figure 1: Example of a mirror graph: cubic permutahedron with corresponding hyperplane arrangement

graph G defined as follows: $ab\Theta xy$ if $d(a, x) + d(b, y) \neq d(a, y) + d(b, x)$, where d is the shortest path distance function. Thus a mirror partition, if it exists, is unique and easily computable based on the metric of the graph. It will follow from our results that also mirror automorphisms can be recognized efficiently.

Since partial cubes have inherent metric properties, the connection described above can be used to better understand both classes of graphs. In fact, one of the motivations of mirror graphs was to build examples of cubic partial cubes – intriguing class of graphs with many surprising properties. Nevertheless, the connection does not directly explain geometric properties exposed in examples of mirror graphs of [4]. Even vertex-transitivity of mirror graphs is not a characterizing properties since vertex-transitive partial cubes are only classified in the cubic case [14] and not all of them are mirror graphs. In this paper we will expose a connection between mirror graphs and (*realizable*) *oriented matroids*, explaining the geometric properties of the former. For a general definition of oriented matroids and their connections to graphs see [2, 3]; in this paper we will limit ourselves to realizable oriented matroids, for the sake of simplicity.

One way to describe them is to consider a set of $m > 0$ pairwise different hyperplanes $\{H_1, \dots, H_m\}$ in \mathbb{R}^n , for some $n > 0$, all incident with the origin of the space. Such a *hyperplane arrangement* cuts \mathbb{R}^n into connected spaces called *chambers*. The *tope graph* of a hyperplane arrangement is a graph whose vertices are chambers and two chambers are adjacent if they are separated by a single hyperplane. Tope graphs can be isometrically embedded in hypercubes [3, Proposition 4.2.3], but the reverse problem of characterizing graphs which are tope graphs of hyperplane arrangements is an open problem. A more broad problem of characterizing graphs which correspond to arrangements of pseudohyperplanes in a real projective space is equivalent to characterizing the tope graphs of oriented matroids and was answered in [7].

For an example consider the following classical problem. For a hyperplane H_i with an orthogonal vector v_i its *reflection* is the map $\sigma_{H_i}(x) = x - 2 \frac{x \cdot v_i}{v_i \cdot v_i} v_i$. Arrangements of hyperplanes $\{H_1, \dots, H_m\}$ in \mathbb{R}^n such that all hyperplanes include vector 0, and for every $i \in \{1, \dots, m\}$ the re-

flection of H_i permutes the hyperplanes $\{H_1, \dots, H_m\}$ are called *reflection arrangements*. A simple example is a collection of m vectors in a plane such that the angle between v_i and a chosen axis is $\frac{i}{\pi}$, while for a more complicated example see Figure 1.

A *Coxeter group* is a group which can be presented by generators and relations as $\langle \alpha_1, \dots, \alpha_m \mid (\alpha_i \alpha_j)^{k_{ij}} = 1 \text{ for all } 1 \leq i, j \leq m \rangle$, where $k_{ii} = 1$ and $k_{ij} \geq 2$ for all $0 \leq i < j \leq m$. By a classical result [3, Theorem 2.3.7], the reflection arrangements are in one to one correspondence with the finite Coxeter groups since the tope graphs of the reflection arrangements are the Cayley graphs of the finite Coxeter groups and vice versa. Moreover the finite Coxeter groups were classified by Coxeter [6]. They give rise to four infinite families and six exceptional cases of irreducible reflection arrangements. Here irreducible means that there is no non-trivial partition of hypercubes in two mutually orthogonal classes; equivalently, their tope graphs are not the Cartesian product of smaller tope graphs.

In this paper we characterize mirror graphs as precisely the Cayley graphs of the finite Coxeter groups and thus the tope graphs of reflection arrangements (see Theorem 2.8). This not just fully classifies mirror graphs, but also gives a graph theoretical characterization of reflection arrangements, a subproblem of a more general problem described above, and a characterization of the Cayley graphs of the finite Coxeter groups. Moreover, we provide a polynomial algorithm for the recognition of the three coinciding classes. The rest of the paper is organized as follows. First we give some basic results, definitions and notations needed throughout the paper, while in the next section we prove the main result asserted in Theorem 2.8.

First a few simple results about partial cubes. Relation Θ [8, 17] as defined above is an equivalence relation in partial cubes – we will write F_{uv} for the set of all edges that are in relation Θ with uv . For an edge uv in G define W_{uv} as the subset of vertices of G that are closer to vertex u than to v , that is $W_{uv} = \{w : d(u, w) < d(v, w)\}$. Notice that in a mirror graph G sets W_{uv}, W_{vu} coincide with sets of vertices of G_i^1 and G_i^2 , where i is the index of E_i for which $E_i = F_{uv}$. For the sake of consistency we will prefer the partial cubes notation over the mirror graphs notation. We will write U_{uv} for the subset of vertices in W_{uv} which have a neighbor in W_{vu} . A path P in a partial cube G is a shortest path if and only if it has all of its edges in pairwise different Θ -classes. For fixed u, v , all shortest u, v -paths pass the same Θ -classes of G . If C is a closed walk, then C passes each Θ -class an even number of times. For more information on the relation Θ , we refer the reader to [11, Chapter 11].

We say that a subgraph H of G is *convex* if all the shortest paths in G between vertices in H lie in H . In [15] a *convex traverse* was introduced as follows: Let $v_1 u_1 \Theta v_2 u_2$ in a partial cube G , with $v_2 \in U_{v_1 u_1}$. Let C_1, \dots, C_n , $n \geq 1$, be a sequence of convex cycles such that $v_1 u_1$ lies only on C_1 , $v_2 u_2$ lies only on C_n , and each pair C_i and C_{i+1} , for $i \in \{1, \dots, n-1\}$, intersects in exactly one edge and this edge is in $F_{v_1 u_1}$, all the other pairs do not intersect. If the shortest path from v_1 to v_2 on the union of C_1, \dots, C_n is a shortest v_1, v_2 -path in G , then we call C_1, \dots, C_n a convex traverse from $v_1 u_1$ to $v_2 u_2$. In this case the shortest path from u_1 to u_2 on the union of C_1, \dots, C_n is also a shortest path and we call this two paths the *sides* of a traverse. Most importantly, it was proved in [15] that for arbitrary edges $v_1 u_1, v_2 u_2$ with $v_1 u_1 \Theta v_2 u_2$ there exists a convex traverse connecting them.

One of the concepts closely connected to oriented matroids is *antipodality*. Call a graph G *even* if every vertex v of G has a unique *antipodal* vertex \bar{v} at the distance $\text{diam}(G)$ from v . Moreover,

if for every adjacent vertices u, v , also antipodes \bar{u}, \bar{v} are adjacent, then call G *harmonic-even*. It follows from [10] and [13, Proposition 3.1] that G is harmonic-even partial cube if and only if every vertex v has a vertex \bar{v} at the distance $i(G)$ from v , where $i(G)$ denotes the *isometric dimension* of G , i.e. the number of Θ -classes in G .

We will consider the right actions of groups on the vertices of graphs and for an element α of a group A acting on a graph G we will denote by v^α the image of a vertex v in G by the action of α . We will denote the Cayley graph of a group A , with generators $S = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$ such that $S^{-1} = S$ and $1 \notin S$, with $\text{Cay}(A, S)$, and interpret it as the graph with vertex set A and two elements $\beta_1, \beta_2 \in A$ adjacent if and only if $\beta_1 = \alpha_i \beta_2$ for some $i \in \{1, \dots, k\}$.

2. Results

We start by exposing a crucial property of mirror graph from which many properties will follow.

Lemma 2.1. *A mirror graph G is harmonic-even.*

Proof. As mentioned in the preliminaries, it is enough to prove that every vertex $v \in G$ has a vertex v' at the distance $i(G)$ from v , where $i(G)$ is the isometric dimension of G . Choose an arbitrary $v \in V(G)$ and let u be a vertex that is at the maximal distance from v . By vertex transitivity, $d(v, u) = \text{diam}(G)$. For the sake of contradiction assume that $d(v, u) < i(G)$, thus there exists a Θ -class, say F_{ab} , such that $v, u \in W_{ab}$. Let α_{ab} be a mirror automorphism of G that maps W_{ab} to W_{ba} with mapping every element of U_{ab} to its neighbor.

Let P be a shortest $v^{\alpha_{ab}}, u$ -path. Since F_{ab} is a cut and $v^{\alpha_{ab}} \in W_{ba}, u \in W_{ab}$, there exists an edge $a'b'$ on P that is in relation Θ with ab , where $a' \in U_{ab}$. The automorphism α_{ab}^{-1} maps the subpath of P connecting $v^{\alpha_{ab}}$ and b' to a path connecting v and a' . The union of this path and the subpath of P connecting a' and u is a v, u -path of length $d(u, v^{\alpha_{ab}}) - 1$. Thus $d(v, u) \leq d(v^{\alpha_{ab}}, u) - 1 \leq \text{diam}(G) - 1$. A contradiction. \square

Lemma 2.2. *Let G be a harmonic-even partial cube, and F_{ab}, F_{cd} its arbitrary Θ -classes. Then there exists a convex cycle in G that includes edges from F_{ab} and F_{cd} .*

Proof. Let vu be an arbitrary edge in F_{ab} and \bar{v}, \bar{u} the antipodal vertices of v, u , respectively. Since G is harmonic-even, \bar{u} is adjacent to \bar{v} . Moreover, if $u \in W_{ba}$ and $v \in W_{ab}$, then $\bar{u} \in W_{ab}$ and $\bar{v} \in W_{ba}$ since the antipode of a vertex in a harmonic-even partial cube is at the distance $i(G)$ from it. Hence $\bar{u}\bar{v} \in F_{ab}$. Let T be a convex traverse from vu to $\bar{u}\bar{v}$. Since $d(v, \bar{v}) = i(G)$, all the Θ -classes of G cross T , thus there exists a convex cycle on T that includes edges from F_{cd} . By definition of a traverse, the cycle also includes edges from F_{ab} . \square

For the next lemma notice the following. Let α be an automorphism of a partial cube G . Relation Θ is defined based on distances in a graph, thus automorphisms of G map Θ -classes to Θ -classes. Assume that we know for each Θ -class of G onto which Θ -class it is mapped by α , and additionally an image v^α of some vertex v of G . Let u be an arbitrary vertex of G and P a v, u -path. The knowledge of α determines where path P is mapped by α , since the beginning is mapped to v^α and each vertex is incident with at most one edge of each Θ -class. In particular, the image of u is fixed. Hence, the assumed knowledge completely determines α .

Lemma 2.3. *Let G be a harmonic-even partial cube and F_{ab} a Θ -class in G . If α is an automorphism such that for each edge $vu \in F_{ab}$ it holds that $v^\alpha = v$ and $u^\alpha = u$, then α is the identity map.*

Proof. Let α be an automorphism of G as described in the assertion. By the notice before the lemma and the fact that $v^\alpha = v$ for an arbitrary $v \in U_{ab}$, it suffice to show that every Θ -class of G is mapped to itself by α . By definition, this holds for F_{ab} . On the other hand, if F_{cd} is any Θ -class in G , different from F_{ab} , let C be a convex cycle that includes edges from F_{ab} and F_{cd} , provided by Lemma 2.2. Since C is convex, it has two antipodal edges in F_{ab} . All four endpoints of the edges are mapped to itself by α . Since C is convex, this implies that all the vertices on C must be mapped to itself. In particular, edges from F_{cd} that lie on C are mapped to itself, thus also F_{cd} is mapped to itself. \square

Corollary 2.4. *For a harmonic-even partial cube G and a Θ -class F_{ab} in G , there exists at most one automorphism α_{ab} of G such that for each $uv \in F_{ab}$ it holds $v^{\alpha_{ab}} = u$ and $u^{\alpha_{ab}} = v$. Moreover, $\alpha_{ab}^2 = 1$.*

Proof. Let α_{ab} be an automorphism from the assertion. For the automorphism α_{ab}^2 it holds that $v^{\alpha_{ab}^2} = v$, $u^{\alpha_{ab}^2} = u$ for every $vu \in F_{ab}$. Thus $\alpha_{ab}^2 = 1$, by Lemma 2.3. If α' is an arbitrary automorphism of G that maps every element of U_{ab} to its neighbor in U_{ba} and vice versa, then also for $\alpha_{ab}\alpha'$ holds that $v^{\alpha_{ab}\alpha'} = v$ and $u^{\alpha_{ab}\alpha'} = u$ for all $vu \in F_{ab}$. This implies $\alpha' = \alpha_{ab}^{-1} = \alpha_{ab}$. \square

If G is a mirror graph and xy an edge in G , we can, due to Lemma 2.1 and Corollary 2.4, denote with α_{xy} the unique *mirror automorphism* of G : the automorphism that maps each vertex in U_{xy} to its neighbor in U_{yx} and vice versa. The uniqueness of it leads to the following polynomial algorithm that for a graph G with n vertices and m edges decides if G is a mirror graph, and in the positive case outputs its mirror partition and mirror automorphisms:

Algorithm 1 Recognition of mirror graphs

1. First check if G is a partial cube by calculating the Θ -classes and obtaining its embedding in a hypercube. This can be done in $O(n^2)$ by [9]. The Θ -classes are candidates for the mirror partition of G . If G is not a partial cube, it is not a mirror graph.
 2. For each Θ -class F_{ab} , its corresponding mirror automorphism α_{ab} , if existent, must map all the convex cycles crossed by F_{ab} to themselves. By Lemma 2.2 this determines the image of each Θ -class, and thus gives a candidate for the mirror automorphism. Convex cycles of G can be found in $O(mn^2)$ by [9], obtaining at most $O(nm)$ of them by [1]. Iterating through convex cycles we can determine for each Θ -class how its corresponding mirror automorphism permutes the other Θ -classes.
 3. Considering G embedded in a hypercube, each permutation of Θ -classes can be seen as a permutation of coordinates of the hypercube that G is embedded into, and thus it can easily be checked if the candidates for the mirror automorphisms in fact define automorphisms of G . If so, we output the Θ -classes and the corresponding mirror automorphisms.
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Lemma 2.5. *Let G be a mirror graph, α_{xy} an arbitrary mirror automorphism, and let $v' = v^{\alpha_{xy}}$ for a chosen $v \in V(G)$. Then there exists a path $P = vv_1v_2 \dots v_{n-1}v'$ from v to v' , such that $\alpha_{xy} = \alpha_{vv_1}\alpha_{v_1v_2}\alpha_{v_2v_3} \dots \alpha_{v_{n-1}v'}$.*

Proof. We will prove the lemma by induction on the distance from v to F_{xy} , i.e. the distance from v to the closest edge that is in F_{xy} . It clearly holds if v is incident with F_{xy} , i.e. the distance is 0. Assume that the distance from v to F_{xy} is $d > 0$, and that the lemma holds for all the vertices at the distance less than d . Take an arbitrary shortest path connecting v and F_{xy} and let u be the neighbor of v on it. Let $u' = u^{\alpha_{xy}}$. Notice that α_{xy} maps F_{vu} to $F_{v'u'}$, by Corollary 2.4 it also maps $F_{v'u'}$ to F_{vu} . By the induction assumption, there exists a path $P' = uu_1u_2 \dots u_{n-1}u'$ from u to u' , such that $\alpha_{xy} = \alpha_{uu_1}\alpha_{u_1u_2}\alpha_{u_2u_3} \dots \alpha_{u_{n-1}u'}$. Therefore, to prove the lemma, it is enough to prove that $\alpha_{xy} = \alpha_{vu}\alpha_{xy}\alpha_{u'v'}$.

We will prove that $\alpha_{vu}\alpha_{xy}\alpha_{u'v'}\alpha_{xy} = 1$. Consider the image of F_{vu} by $\alpha_{vu}\alpha_{xy}\alpha_{u'v'}\alpha_{xy}$. Automorphism α_{vu} maps F_{vu} to itself, α_{xy} maps F_{vu} to $F_{v'u'}$, by the above notice, $\alpha_{u'v'}$ maps $F_{v'u'}$ to itself, while α_{xy} maps $F_{v'u'}$ to F_{uv} . Thus $\alpha_{vu}\alpha_{xy}\alpha_{u'v'}\alpha_{xy}$ maps F_{uv} to itself. Now take an arbitrary $w \in U_{uv}$, let z be its neighbour in U_{vu} , and denote with $w' = w^{\alpha_{xy}}$, $z' = z^{\alpha_{xy}}$. Notice that $w'z' \in F_{v'u'}$. Then α_{vu} maps w to z , α_{xy} maps z to z' , $\alpha_{u'v'}$ maps z' to w' , while α_{xy} maps w' to w . By Lemma 2.3 and the fact that w was an arbitrary element of U_{uv} , it follows that $\alpha_{vu}\alpha_{xy}\alpha_{u'v'}\alpha_{xy} = 1$. This finishes the proof. \square

Lemma 2.6. *Let $C = (v_0v_1 \dots v_{2i-1})$ be a convex cycle in a mirror graph G . For every $j \in \{0, 1, \dots, 2i-1\}$ it holds*

$$\alpha_{v_{j+1}v_j} = \alpha_{v_1v_0}(\alpha_{v_0v_{2i-1}}\alpha_{v_1v_0})^j.$$

Moreover

$$\alpha_{v_0v_1}\alpha_{v_1v_2} \dots \alpha_{v_{2i-1}v_0} = \alpha_{v_0v_{2i-1}}\alpha_{v_{2i-1}v_{2i-2}} \dots \alpha_{v_1v_0} = 1.$$

Proof. Since C is a convex cycle, it has its antipodal pairs of edges in relation Θ . The latter implies that every mirror automorphism of a Θ -class with edges on C maps C to C . We will first prove that $\alpha_{v_{j+1}v_j} = \alpha_{v_jv_{j-1}}\alpha_{v_{j-1}v_{j-2}}\alpha_{v_jv_{j-1}}$ for every $j \in \mathbb{Z}_{2i}$. Let xy be an arbitrary edge from $F_{v_{j+1}v_j}$. Since $\alpha_{v_jv_{j-1}}$ maps C to itself, it must map $v_{j+1}v_j$ to $v_{j-2}v_{j-1}$, thus it also maps $F_{v_{j+1}v_j}$ to $F_{v_{j-1}v_{j-2}}$. This implies that $\alpha_{v_jv_{j-1}}$ maps edge xy to an edge in $F_{v_{j-1}v_{j-2}}$. Say $w = x^{\alpha_{v_jv_{j-1}}}$ and $z = y^{\alpha_{v_jv_{j-1}}}$. By definition of a mirror automorphism, $\alpha_{v_{j-1}v_{j-2}}$ maps w to z and vice versa. Thus $\alpha_{v_jv_{j-1}}\alpha_{v_{j-1}v_{j-2}}\alpha_{v_jv_{j-1}}$ maps the pair (x, y) to the pair (y, x) for every edge $xy \in F_{v_{j+1}v_j}$. By Corollary 2.4, this automorphism must equal $\alpha_{v_{j+1}v_j}$.

Now we prove that $\alpha_{v_{j+1}v_j} = \alpha_{v_1v_0}(\alpha_{v_0v_{2i-1}}\alpha_{v_1v_0})^j$. We will prove the assertion by induction on j . It clearly holds for $j = 0$, while for $j = 1$ we have $\alpha_{v_2v_1} = \alpha_{v_1v_0}\alpha_{v_0v_{2i-1}}\alpha_{v_1v_0}$ by the previous paragraph. We calculate:

$$\begin{aligned} \alpha_{v_{j+1}v_j} &= \alpha_{v_jv_{j-1}}\alpha_{v_{j-1}v_{j-2}}\alpha_{v_jv_{j-1}} \\ &= \alpha_{v_1v_0}(\alpha_{v_0v_{2i-1}}\alpha_{v_1v_0})^{j-1}\alpha_{v_1v_0}(\alpha_{v_0v_{2i-1}}\alpha_{v_1v_0})^{j-2}\alpha_{v_1v_0}(\alpha_{v_0v_{2i-1}}\alpha_{v_1v_0})^{j-1} \\ &= \alpha_{v_1v_0}\alpha_{v_0v_{2i-1}}(\alpha_{v_1v_0}\alpha_{v_0v_{2i-1}})^{j-2}\alpha_{v_1v_0}\alpha_{v_1v_0}(\alpha_{v_0v_{2i-1}}\alpha_{v_1v_0})^{j-2}\alpha_{v_1v_0}(\alpha_{v_0v_{2i-1}}\alpha_{v_1v_0})^{j-1} \\ &= \alpha_{v_1v_0}(\alpha_{v_0v_{2i-1}}\alpha_{v_1v_0})^j \end{aligned}$$

Notice that the latter implies that $\alpha_{v_0 v_{2i-1}} = \alpha_{v_i v_{i-1}} = \alpha_{v_1 v_0} (\alpha_{v_0 v_{2i-1}} \alpha_{v_1 v_0})^{i-1}$ (first equation holds since antipodal edges on C are in relation Θ), thus

$$1 = (\alpha_{v_0 v_{2i-1}} \alpha_{v_1 v_0})^i. \quad (1)$$

Let A_C be the subgroup of automorphisms of G generated by the mirror automorphisms $\alpha_{v_1 v_0}, \dots, \alpha_{v_{2i} v_{2i-1}}$. We have proved that $\alpha_{v_0 v_{2i-1}}$ and $\alpha_{v_1 v_0}$ generate A_C . Since (1) holds, A_C must be a quotient of the group $\langle \alpha_{v_0 v_1}, \alpha_{v_0 v_{2i-1}} \mid (\alpha_{v_0 v_1} \alpha_{v_0 v_{2i-1}})^i = 1, \alpha_{v_0 v_1}^2 = 1, \alpha_{v_0 v_{2i-1}}^2 = 1 \rangle$, which is a Coxeter group of order $2i$. Since A_C acts transitively on C , it must have at least $2i$ elements. Thus A_C is isomorphic to $\langle \alpha_{v_0 v_1}, \alpha_{v_0 v_{2i-1}} \mid (\alpha_{v_0 v_1} \alpha_{v_0 v_{2i-1}})^i = 1, \alpha_{v_0 v_1}^2 = 1, \alpha_{v_0 v_{2i-1}}^2 = 1 \rangle$ which is precisely the group of all edge symmetries of C . In particular it holds,

$$\alpha_{v_0 v_1} \alpha_{v_1 v_2} \cdots \alpha_{v_{2i-1} v_{2i}} = \alpha_{v_0 v_{2i-1}} \alpha_{v_{2i-1} v_{2i-2}} \cdots \alpha_{v_1 v_0} = 1.$$

□

Lemma 2.7. *Let $C = (v_0 v_1 \dots v_{2i-1})$ be a convex cycle in a mirror graph. Then any edge $ab \in F_{v_0 v_1}$ is in the intersection of $W_{v_2 v_1}, W_{v_3 v_2}, \dots, W_{v_i v_{i-1}}$ or in the intersection of the complements of these sets.*

Proof. Assume that $ab \in F_{v_0 v_1}$ is in $W_{v_2 v_1}$. Consider a convex traverse from $v_0 v_1$ to ab . Since $v_0, v_1 \in W_{v_1 v_2}$, $a, b \in W_{v_2 v_1}$ and $F_{v_1 v_2}$ is a cut, there must be a convex cycle D on T , that includes an edge from $F_{v_1 v_2}$. Pick an arbitrary $v_{j-1} v_j$ for $3 \leq j \leq i$. Then for $k = \lfloor j/2 \rfloor$, $\alpha_{v_k v_{k+1}}$ maps $v_{j-1} v_j$ to $v_0 v_1$ or $v_1 v_2$ (and vice versa) since it maps C to C . Hence it maps $F_{v_0 v_1}$ or $F_{v_1 v_2}$ to $F_{v_{j-1} v_j}$. On the other hand, by Lemma 2.6, $\alpha_{v_k v_{k+1}}$ can be expressed as a combination of automorphisms $\alpha_{v_0 v_1}$ and $\alpha_{v_1 v_2}$. Since each such automorphism maps D to D , also $\alpha_{v_k v_{k+1}}$ maps D to D . Thus there must be an edge on D in $F_{v_{j-1} v_j}$. This implies $a, b \in W_{v_j v_{j-1}}$.

On the other hand, $ab \in F_{v_0 v_1}$ can lie in $W_{v_1 v_2} = W_{v_{i+2} v_{i+1}}$. Now the result follows if we consider a traverse from $v_i v_{i+1}$ to ab and use the same arguments. □

One could easily deduce from Lemma 2.7, using results from [12], that every mirror graph is in fact a tope graph of an oriented matroid. We will not make this argumentation since this result will follow from the main theorem.

Denote with $C(G)$ the 2-dimensional cell complex whose 2-cells are obtained by replacing each convex cycle C of length $2j$ of G by a regular Euclidean polygon $[C]$ with $2j$ sides. In [5] it was proved that for a partial cube G , the complex $C(G)$ is simply connected.

Theorem 2.8. *For a graph G the following statements are equivalent:*

- (i) G is a mirror graph.
- (ii) G is the Cayley graph of a finite Coxeter group.
- (iii) G is the tope graph of a reflection arrangement.

Proof. The crucial part is to prove that (i) \implies (ii). Assume that G is a mirror graph. We will first prove that G is a Cayley graph. Let A be the subgroup of $\text{Aut}(G)$ generated by all the mirror automorphisms. Group A acts transitively on the vertices of G . Recall that by a theorem of

Sabidussi [16], G is a Cayley graph of a group A if A acts transitively on the vertices of G and the stabilizers of the vertices are trivial. Therefore, to prove the assertion it suffice to prove that for an arbitrary vertex $v \in V(G)$ its stabilizer is trivial. Assume that an automorphism $\alpha_{a_1 b_1} \alpha_{a_2 b_2} \dots \alpha_{a_n b_n}$ maps v to itself. Let $v_1 = v^{\alpha_{a_1 b_1}}$, $v_2 = v_1^{\alpha_{a_2 b_2}}$, \dots , $v = v_{n-1}^{\alpha_{a_n b_n}}$. By Lemma 2.5, $\alpha_{a_1 b_1} \alpha_{a_2 b_2} \dots \alpha_{a_n b_n}$ equals to $\alpha_{vu_1} \alpha_{u_1 u_2} \dots \alpha_{u_{m-1} v}$ where $(vu_1 u_2 u_3 \dots u_{m-1})$ is a closed walk from v to v passing v_1, v_2, \dots, v_{m-1} . Graph G is a partial cube, hence its 2-dimensional cell complex $C(G)$ made out of the convex cycles is simply connected. Since for each convex cycle $(a_0 a_1 \dots a_{2i-1})$ in G , by Lemma 2.6, holds that $a_{a_0 a_1} a_{a_1 a_2} \dots a_{a_{2i-1} a_0} = 1$, the latter implies that we can transform $\alpha_{vu_1} \alpha_{u_1 u_2} \dots \alpha_{u_{m-1} v}$ to the identity using equalities on convex cycles. Thus the stabilizer of v is trivial, and G is a Cayley graph.

Now pick an arbitrary vertex $v \in V(G)$ and let vv_1, vv_2, \dots, vv_k be the edges incident with it. Then $G \cong \text{Cay}(A, \{\alpha_{vv_1}, \dots, \alpha_{vv_k}\})$. We want to identify group A to understand the structure of G . First identify vertices of G with elements of A in the standard way: identify the chosen vertex v with the identity 1 of A and every vertex u of G with the unique automorphism $\alpha \in A$ such that $u = v^\alpha$. By definition of the Cayley graph, every relation of generators, say $\alpha_{vv_{i_1}} \dots \alpha_{vv_{i_2}} \alpha_{vv_{i_1}} = 1$, gives us a closed walk on vertices $1, \alpha_{vv_{i_1}}, \alpha_{vv_{i_2}} \alpha_{vv_{i_1}}, \alpha_{vv_{i_3}} \alpha_{vv_{i_2}} \alpha_{vv_{i_1}}, \dots, \alpha_{vv_{i_{p-1}}} \dots \alpha_{vv_{i_2}} \alpha_{vv_{i_1}}$. For the latter we will say that a relation in A generates the closed walk in G . First we prove the following claim:

Claim 1. *Every pair of incident edges vv_i, vv_j lies on the unique convex cycle C generated by the relation $(\alpha_{vv_i} \alpha_{vv_j})^{k_{ij}} = 1$, where $k_{ij} \geq 2$ equals half of the length of C .*

Proof. Let C be the closed walk generated by the relation $(\alpha_{vv_i} \alpha_{vv_j})^{k_{ij}} = 1$, where k_{ij} is as small as possible (k_{ij} exists since A is finite). First we want to identify Θ -classes of edges that lie on C . Let D be a convex cycle in G with nontrivial intersection with F_{vv_i} and F_{vv_j} , provided by Lemma 2.2. Denote with xy an edge on D that is F_{vv_i} , and without loss of generality assume that D and xy are such that the distance between vv_i and xy is as small as possible. There are two antipodal edges on D that are in F_{vv_j} , let wz be the one that is in W_{vv_i} .

Now we prove that xy and wz are incident. Let T_i be a convex traverse connecting vv_i and xy , and T_j a convex traverse connecting vv_j and wz . There is no edge in F_{vv_j} on T_i since otherwise there would exist a convex cycle with a nontrivial intersection with F_{vv_i} and F_{vv_j} but closer to vv_i than D . Also, there is no edge in F_{vv_i} on T_j since $vv_j, wz \in W_{vv_i}$. Assume that there is an edge ab on the shortest path connecting xy and wz on D , say $xy \in W_{ab}$. Then the sides of T_i and T_j , together with the shortest path connecting xy and wz form a closed walk. Since a closed walk must pass each Θ -class even number of times, this implies that one of T_i, T_j has an edge in F_{ab} . Without loss of generality assume that T_i has an edge in F_{ab} . In this case the edge vv_i is in W_{ba} , but not in W_{vj} . A contradiction with Lemma 2.7. Thus xy and wz are incident.

Denote with $(u_0 u_1 \dots u_{2k-1})$ the vertices of D , where $wz = u_0 u_1$, $xy = u_0 u_{2k-1}$ and $2k$ is the length of D . First notice that $\alpha_{vv_j} = \alpha_{u_0 u_1}$ and $\alpha_{vv_i} = \alpha_{u_0 u_{2k-1}}$ since $vv_j \Theta u_0 u_1$ and $vv_i \Theta u_0 u_{2k-1}$. By Lemma 2.6, the smallest k_{ij} such that $(\alpha_{u_0 u_{2k-1}} \alpha_{u_0 u_1})^{k_{ij}} = 1$ is k , i.e. k_{ij} is half the length of D . Thus also C has length $2k$.

Edges on C are connecting vertices of the form $(\alpha_{vv_i} \alpha_{vv_j})^l$ and $\alpha_{vv_j} (\alpha_{vv_i} \alpha_{vv_j})^l$, or $\alpha_{vv_j} (\alpha_{vv_i} \alpha_{vv_j})^l$ and $(\alpha_{vv_i} \alpha_{vv_j})^{l+1}$, for some $0 \leq l < k$. Automorphism $\alpha_{vv_j} (\alpha_{vv_i} \alpha_{vv_j})^l$ (by the right action) maps vertex $(\alpha_{vv_i} \alpha_{vv_j})^l$ to α_{vv_j} and vertex $\alpha_{vv_j} (\alpha_{vv_i} \alpha_{vv_j})^l$ to 1. But $\alpha_{vv_j} (\alpha_{vv_i} \alpha_{vv_j})^l = \alpha_{u_0 u_1} (\alpha_{u_0 u_{2k-1}} \alpha_{u_0 u_1})^l$, and,

by Lemma 2.6, $\alpha_{u_0 u_1}(\alpha_{u_0 u_{2k-1}} \alpha_{u_0 u_1})^l = \alpha_{u_{l+1} u_l}$. Automorphism $\alpha_{u_{l+1} u_l}$ maps $F_{u_{2l+1} u_{2l}}$ to $F_{u_0 u_1}$. Since the edge between $(\alpha_{vv_i} \alpha_{vv_j})^l$ and $\alpha_{vv_j}(\alpha_{vv_i} \alpha_{vv_j})^l$ is mapped to edge between 1 and vv_j which is in $F_{u_0 u_1}$, it must be in $F_{u_{2l+1} u_{2l}}$.

Similar analysis can be made for the edges between $\alpha_{vv_j}(\alpha_{vv_i} \alpha_{vv_j})^l$ and $(\alpha_{vv_i} \alpha_{vv_j})^{l+1}$, for $0 \leq l < k$. Thus all the edges on C are in the same Θ -classes as edges on D . Now we can prove that C is convex. Let ab be an edge on C different from vv_i but in F_{vv_i} . Consider a convex traverse T from vv_i to ab . Since there is a path on C connecting both edges that includes only edges that are in Θ relation with edges on D , we deduce that also all the edges on T are in Θ relation with edges on D . By Lemma 2.7, it follows that T must be a single convex cycle E of length the same as D , that is $2k$.

For the uniqueness, we shall prove that no two convex cycles can share more than an edge or a vertex. Assume that two different convex cycles D_1, D_2 share two vertices. Since they are convex, they share a shortest path connecting this two vertices. Now, assume that there are at least two edges on this path. Let $x_0 x_1 x_2$ be a subpath that is shared by both, such that x_0 is also incident with non-identical edges $x_0 y_1$ and $x_0 y_2$, such that they lie on $D_1 \setminus D_2$ and $D_2 \setminus D_1$, respectively. The mirror automorphism $\alpha_{x_0 x_1}$ maps D_1 onto D_1 and D_2 onto D_2 . But this is impossible since $x_1 x_2$ cannot simultaneously get mapped to $x_0 y_1$ and $x_0 y_2$. \square

We claim that $A = \langle \alpha_{vv_1}, \alpha_{vv_2}, \dots, \alpha_{vv_k} \mid (\alpha_{vv_i} \alpha_{vv_j})^{k_{ij}} = 1 \rangle$, where $k_{ii} = 1$ and $k_{ij} \geq 2$ is given by Claim 1. Since G is a connected Cayley graph of A , A is generated by $\alpha_{vv_1}, \alpha_{vv_2}, \dots, \alpha_{vv_k}$ and the relations hold by Claim 1 and Lemma 2.6. Assume that for the above generators some other relation holds, say $\alpha_{vx_1} \alpha_{vx_2} \dots \alpha_{vx_j} = 1$ for $x_i \in \{v_1, \dots, v_k\}$ for all $i \in \{1, \dots, j\}$. We want to prove that this relation can be derived from the above relations. Again identify vertices of G with elements of A in the standard way.

Claim 2. *Let D be a convex cycle in G . Then D lies on vertices $\alpha_{vv_i} a, \alpha_{vv_j} \alpha_{vv_i} a, \alpha_{vv_i} \alpha_{vv_j} \alpha_{vv_i} a, \dots, (\alpha_{vv_j} \alpha_{vv_i})^{k_{ij}} a$, for some generators $\alpha_{vv_i}, \alpha_{vv_j}$ and some $a \in A$.*

Proof. Pick a vertex u on D and let $a \in A$ be the map that maps v to u , i.e. u is identified with a . Let $\alpha_{vv_i} a, \alpha_{vv_j} a$ be the neighbors of a on D . Then automorphism $a^{-1} \in A$ maps D to a convex cycle C incident with 1, $\alpha_{vv_i}, \alpha_{vv_j}$. By Claim 1 the only such cycle C is the cycle on vertices $\alpha_{vv_i}, \alpha_{vv_j} \alpha_{vv_i}, \alpha_{vv_i} \alpha_{vv_j} \alpha_{vv_i}, \dots, (\alpha_{vv_j} \alpha_{vv_i})^{k_{ij}}$. Automorphism a maps C to D which proves the claim. \square

Assume that the relation $\alpha_{vx_1} \alpha_{vx_2} \dots \alpha_{vx_j} = 1$ holds in A . As above it generates a closed walk in G . Since G is a partial cube, its 2-dimensional cell complex $C(G)$ is simply connected, thus every closed walk is generated by convex cycles in G . In the language of the generators this implies that all the relations in G can be derived from relations on the convex cycles of G . By Claim 2, the relations on the convex cycles of G are derived from asserted relations.

We have proved that $A = \langle \alpha_{vv_1}, \alpha_{vv_2}, \dots, \alpha_{vv_k} \mid (\alpha_{vv_i} \alpha_{vv_j})^{k_{ij}} = 1 \rangle$. Thus A is a finite Coxeter group and (i) \implies (ii) follows. As described in the preliminaries, the Cayley graphs of the finite Coxeter groups are in one to one correspondence with the tope graphs of the reflection arrangements thus (ii) \iff (iii).

Finally, assume that G is the tope graph of a reflection arrangement. Then we can partition the edges of G into sets corresponding to the hyperplanes in the arrangement. Moreover, the reflection

of each hyperplane maps chambers to chambers and hyperplanes to hyperplanes, and thus induces a mirror automorphism of G . We deduce that G is a mirror graph and (iii) \implies (i) follows. \square

Theorem 2.8 implies that via Algorithm 1 also the Cayley graphs of the finite Coxeter groups and the tope graphs of the reflection arrangements can be recognized in polynomial time.

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References

References

- [1] J. Azarija and S. Klavžar. Moore graphs and cycles are extremal graphs for convex cycles. *J. Graph Theory*, 80:34–42, 2015.
- [2] H.-J. Bandelt, V. Chepoi, and K. Knauer. COMs: complexes of oriented matroids. *arXiv preprint arXiv:1507.06111*, 2015.
- [3] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler. *Oriented Matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.
- [4] B. Brešar, S. Klavžar, A. Lipovec, and B. Mohar. Cubic inflation, mirror graphs, regular maps, and partial cubes. *European J. Combin.*, 25:55–64, 2004.
- [5] V. Chepoi, K. Knauer, and T. Marc. Partial cubes without Q_3^- minors. *arXiv preprint arXiv:1606.02154*, 2016.
- [6] H. S. M. Coxeter. The complete enumeration of finite groups of the form $R_1^2 = (R_i R_j)^{k_{ij}} = 1$. *J. London Math. Soc.*, 10:21–25, 1935.
- [7] I. P. F. da Silva. Axioms for maximal vectors of an oriented matroid; a combinatorial characterization of the regions determined by an arrangement of pseudohyperplanes. *European J. Combin.*, 16:125–145, 1995.
- [8] D. Ž. Djoković. Distance-preserving subgraphs of hypercubes. *J. Combin. Theory Ser. B*, 14:263–267, 1973.
- [9] D. Eppstein. Recognizing partial cubes in quadratic time. *J. Graph Algorithms Appl.*, 15:269–293, 2011.
- [10] K. Fukuda and K. Handa. Antipodal graphs and oriented matroids. *Discrete Math.*, 111:245–256, 1993.
- [11] R. Hammack, W. Imrich, and S. Klavžar. *Handbook of Product Graphs*. Discrete Math. Appl. (Boca Raton). CRC Press, 2nd edition, 2011.
- [12] K. Handa. A characterization of oriented matroids in terms of topes. *European J. Combin.*, 11:41–45, 1990.
- [13] S. Klavžar and M. Kovše. On even and harmonic-even partial cubes. *Ars Combin.*, 93:77–86, 2009.
- [14] T. Marc. Classification of vertex-transitive cubic partial cubes. *arXiv preprint arXiv:1509.04565*, 2015.
- [15] T. Marc. There are no finite partial cubes of girth more than 6 and minimum degree at least 3. *European J. Combin.*, 55, 2016.
- [16] G. Sabidussi. On a class of fixed-point-free graphs. *Proc. Amer. Math. Soc.*, 9:800–804, 1958.
- [17] P. M. Winkler. Isometric embedding in products of complete graphs. *Discrete Appl. Math.*, 7:221–225, 1984.